The Dissipation of Energy in Permanent Ocean Currents, with some Relations between Salinities, Temperatures and Currents.

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In a previous paper* the author has discussed the effect of viscosity in dissipating the energy of non-turbulent tidal flow of a sea, and the present one contains a simple extension of the method to non-periodic motion. In practically all parts of the open ocean there exist currents whose directions are almost constant and whose velocities may therefore be supposed to contain a component which is independent of the time. The motion is usually slow, the current seldom exceeding one or two knots, and on the average much less, so that in all probability the effect of turbulence will not be large. The work is therefore carried out on the assumption of a smooth flow, and a first approximation only is made. The results are used to obtain an estimate of the mean rate at which energy is dissipated in all the oceans by these permanent current streams.

The second portion of the paper is devoted to a simple method of deducing the magnitude of the surface current solely from a knowledge of the distribution of the salinity or of the temperature of the water. There are not sufficient data available to obtain exact figures, but rough estimates of the orders of magnitude lead to concordant results.

The Dissipation of Energy in Permanent Ocean Currents.

Consider a portion of the sea so small that in the state of relative equilibrium its surface may be regarded as flat, and let it rotate about the vertical with an angular velocity ω which is to be taken as the component of the earth's diurnal rotation. Let the axis of rotation be taken as the axis of z, the origin being at the bottom of the sea and the equation of the mean surface being $z = h$. Take axes of x, y, rotating in their plane with the prescribed angular velocity ω, and let $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ be the components referred to these axes of the small relative velocity of the particle of water which at time t is at the point (x, y, z).

* 'Roy. Soc. Proc.,' A, vol. 93, p. 348 (1917).

For the horizontal motion the equations are*

$$\left.\begin{aligned}\frac{\partial \mathbf{U}}{\partial t}-2\omega\mathbf{V} &= -g\frac{\partial \mathbf{Z}}{\partial x}+\nu\Delta\mathbf{U}\\ \frac{\partial \mathbf{V}}{\partial t}+2\omega\mathbf{U} &= -g\frac{\partial \mathbf{Z}}{\partial y}+\nu\Delta\mathbf{V}\end{aligned}\right\}, \tag{1}$$

where the density is assumed uniform, ν is the kinematic coefficient of viscosity, Δ is the Laplacian operator, and $\mathbf{Z}-\Pi/g$ is the elevation of the surface above its position of relative equilibrium, Π being the disturbing potential.

Let

$$\left.\begin{aligned}\mathbf{U} &= \mathrm{U}+\Sigma\, u_\sigma \cos(\sigma t-\alpha)\\ \mathbf{V} &= \mathrm{V}+\Sigma\, v_\sigma \cos(\sigma t-\beta)\end{aligned}\right\}, \tag{2}$$

where U, V, u_σ, v_σ are independent of t, and the summation includes terms of different periods. The parts of the solutions which are periodic functions have been discussed previously† on the assumption σ and ω are of the same order of magnitude and that $\sigma-2\omega$ is positive. By a very slight modification the work can be extended to the cases in which σ is less than 2ω, that is, to the longer period motion. We proceed to consider the parts U, V of the solutions which are independent of the time, assuming that ω is not zero.

By equations (1) and (2) these are given by

$$\left.\begin{aligned}-2\omega\mathrm{V} &= -g\frac{\partial Z}{\partial x}+\nu\frac{\partial^2 \mathrm{U}}{\partial z^2}\\ 2\omega\mathrm{U} &= -g\frac{\partial Z}{\partial y}+\nu\frac{\partial^2 \mathrm{V}}{\partial z^2}\end{aligned}\right\}, \tag{3}$$

where Z is the part of $\mathbf{Z}$ independent of the time, and $\Delta(\mathrm{U}, \mathrm{V})$ have been replaced by $\partial^2(\mathrm{U}, \mathrm{V})/\partial z^2$.

Combining these equations we obtain

$$\left(\frac{\partial^2}{\partial z^2}\mp\frac{2\iota\omega}{\nu}\right)(\mathrm{U}\pm\iota\mathrm{V}) = \frac{g}{\nu}\left(\frac{\partial}{\partial x}\pm\iota\frac{\partial}{\partial y}\right)Z,$$

of which the solutions, subject to the conditions U, V zero at the bottom $z=0$, and $\partial\mathrm{U}/\partial z$, $\partial\mathrm{V}/\partial z$ zero at the surface $z=h$, are

$$\mathrm{U}\pm\iota\mathrm{V} = \pm\frac{\iota g}{2\omega}\left[1-\frac{\cosh\{\gamma(h-z)(1\pm\iota)\}}{\cosh\{\gamma h(1\pm\iota)\}}\right]\left(\frac{\partial}{\partial x}\pm\iota\frac{\partial}{\partial y}\right)Z, \tag{4}$$

where $\gamma^2=\omega/\nu$.

In the open sea γh is always very large, so that we can replace these solutions by

$$\mathrm{U}\pm\iota\mathrm{V} = (\mathrm{U}_0\pm\iota\mathrm{V}_0)\left[1-e^{-\gamma z(1\pm\iota)}\right] \tag{5}$$

where U_0, V_0 are the surface values of U, V.

* See, for example, Lamb, 'Hydrodynamics,' §§ 206, 316.

† 'Roy. Soc. Proc.,' A, *loc. cit.*

Now let us suppose that Q is the local magnitude of the non-periodic part of the surface current relative to the earth's rotation, and that the local axes are chosen so that Q is in the direction Ox. Then $U_0 = Q$, $V_0 = 0$, and equations (5) become

$$\left.\begin{aligned} U &= Q(1-e^{-\gamma z}\cos\gamma z) \\ V &= Qe^{-\gamma z}\sin\gamma z \end{aligned}\right\}. \tag{6}$$

It is now easy to justify the neglect of the terms in obtaining equations (3). The non-periodic terms in the equation of continuity lead to the relation

$$\int_0^h \left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right) dz = 0,$$

which from equations (4) is equivalent to

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) Z = 0.$$

Hence from the same equations

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(U, V) = 0,$$

so that the neglected terms are, to this approximation, actually zero.

The total rate of dissipation of energy per unit surface area of the sea may be taken* as

$$-\mu\int_0^h \left(\mathbf{U}\frac{\partial^2\mathbf{U}}{\partial z^2}+\mathbf{V}\frac{\partial^2\mathbf{V}}{\partial z^2}\right) dz$$

where μ is the coefficient of viscosity.

Substituting from equations (2) and taking the mean value over a long period of time we obtain a result which can be expressed in the form

$$-\mu\int_0^h \left(U\frac{\partial^2 U}{\partial z^2}+V\frac{\partial^2 V}{\partial z^2}\right) dz+\Sigma E_\sigma,$$

where E_σ is the contribution from the terms of period $2\pi/\sigma$. The mean rate of energy dissipation due to each term in the current velocity can thus be evaluated separately and the several results added. The contribution from the non-periodic terms given by equations (6) is found to be $\mu\gamma Q^2$ to the order of approximation already used, thus depending only on the velocity and not involving its gradients.

Writing $\omega = \Omega\sin\lambda$, where Ω is the earth's diurnal rotation and λ is the latitude of the place, we obtain for the mean rate, F, of dissipation of energy per unit surface area due to the permanent ocean currents

$$F = \rho Q^2\sqrt{(\nu\Omega\sin\lambda)}, \tag{7}$$

* R. O. Street, *loc. cit.*, p. 352.

where ρ is the density of the water* and ν is the kinematic coefficient of viscosity appropriate to the circumstances.

For places on the equator ω is zero and equations (3) reduce to

$$\frac{\partial^2 U}{\partial z^2} = \frac{g}{\nu}\frac{\partial Z}{\partial x}, \qquad \frac{\partial^2 V}{\partial z^2} = \frac{g}{\nu}\frac{\partial Z}{\partial y},$$

so that, with the same surface and bottom conditions as before, we have in place of (5)

$$\frac{U}{U_0} = \frac{V}{V_0} = \frac{2z}{h} - \frac{z^2}{h^2}.$$

Hence we obtain in place of equation (7)

$$F' = \frac{4}{3}\frac{\mu Q^2}{h}. \tag{8}$$

Now in deep water F' is very small compared with $\rho Q^2 \sqrt{(\nu\Omega)}$. Therefore in estimating the mean rate at which energy is dissipated by the permanent currents of all the oceans we may without appreciable error suppose that equation (7) is valid everywhere.

In order to obtain a numerical estimate for the total actual rate of energy dissipation due to these non-periodic currents, the surface of the sea has been divided into 10-degree "squares" and for each a mean value of Q has been used. These values have been deduced from the Admiralty current charts. At any point the speed of the current is usually variable; for the non-periodic part, the mean of the two extreme values has been taken. Thus when the current is recorded as 20–40 sea miles per day the value 30 has been used. The variations in current strengths are greatest in the neighbourhood of the equator; owing to the presence of the factor $\sin\lambda$ in the formula for the rate of energy dissipation the effect of this variation is not important. In any 10-degree "square" the mean latitude is taken as sufficiently accurate for our present purpose.

The details of the arithmetic are omitted and only the final results for each of the three great oceans given.

For the Atlantic, Pacific, and Indian Oceans the values

$$(7,\ 11,\ 7)\times 10^{17} \text{ ergs per second,} \tag{9}$$

have been obtained by substituting in the formula (7), and assuming that $\nu = 0{\cdot}018$, the areas over which the summation has been carried out being respectively

$$(7{\cdot}3,\ 13{\cdot}5,\ 5{\cdot}4)\times 10^{17} \text{ sq. cm.}$$

* In this portion of the work no appreciable error is introduced by regarding ρ as constant, although in all probability the variations of density actually constitute a primary cause of the currents.

The areas of the surfaces of these oceans are

$$(8{\cdot}2,\ 16{\cdot}6,\ 7{\cdot}3) \times 10^{17} \text{ sq. cm.}$$

but there is little information available beyond the fiftieth parallel of latitude, or the sixtieth in the case of the North Atlantic. Hence we may say that, on the explicit assumption of non-turbulent flow, the presence of viscosity causes the non-periodic parts of the ocean currents to produce a dissipation of energy at the mean rate

$$3 \times 10^{18} \text{ ergs per second.} \tag{10}$$

It may be noted that this result is roughly 1 erg per second per square centimetre of surface. The estimate (10) is to be regarded correct to one figure only.

It is interesting to compare this result with the expression, 22×10^{18} ergs per second, recently obtained by H. Jeffreys* for the mean rate of energy dissipation in the semi-diurnal tidal motion of all the shallow seas of the world. The method there used is due to G. I. Taylor,† and is based on the assumption that the dissipation is caused by direct friction of moving water on the bed of the sea, the velocity of slip being taken equal to the surface current velocity. The rate of energy dissipation is assumed to be expressible in the form‡ $k\rho v^2$ per unit area of bottom, ρ being the density, v the velocity of slip, and k a constant which is taken as equal to that found experimentally for the wind blowing over grass land.§

The Relations between Currents, Salinities, and Temperatures.

An estimate of the magnitude of ocean currents can be obtained by simple methods from a knowledge of the distributions of the salinity or of the temperature of the sea water.

Let s denote the salinity at any point at time t, that is, the mass of salt dissolved in unit mass of the water which at that instant surrounds the point considered. Making the usual assumption that each small portion of water retains its salinity, we have the equation

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0,$$

* 'Phil. Trans.,' A, vol. 221, p. 239 (1920).

† 'Phil. Trans.,' A, vol. 220, p. 1 (1919).

‡ Note that this is not of the same form as F or F′ in equations (7) and (8).

§ There seems to be no assurance that the agreement between this somewhat arbitrary determination of k and another estimate of its value obtained from an extrapolation of Bazin's formula for the flow of water in a river may not be largely accidental. The various hydraulic formulæ for such motion seem to be empirical; and others, which agree equally well with the observations, when extrapolated give results of entirely different orders of magnitude. *Cf.* Lord Rayleigh, 'Sci. Papers,' vol. 6, p. 602.

where the differentiation follows the path of the water particle. If this equation is applied to a particle at the surface of the sea, it can be written in the form

$$\frac{\partial s}{\partial t}+\mathrm{Q}s'\cos\chi+w\frac{\partial s}{\partial z}=0, \tag{11}$$

where Q is the horizontal and w the vertical velocity of the water, the axis of z being measured upwards, s' is the maximum surface gradient of s at time t, and χ is the angle between the directions of Q and s'.

Similarly, if ρ is the density and T the temperature of the water at the point at time t, the assumption that the conduction of heat is negligible leads to the equation

$$\mathrm{D}(\rho\mathrm{T})/\mathrm{D}t=0,$$

while the equation of conservation of mass is

$$\mathrm{D}\rho/\mathrm{D}t=0.$$

From these we deduce

$$\mathrm{DT}/\mathrm{D}t=0,$$

which as before leads to a result of the form

$$\frac{\partial \mathrm{T}}{\partial t}+\mathrm{QT}'\cos\psi+w\frac{\partial \mathrm{T}}{\partial z}=0, \tag{12}$$

where T′ is the maximum surface gradient of T at time t, and ψ is the angle between the directions of Q and T′.

The value of Q could be found from either of the equations (11), (12), if the necessary related sets of observations of the salinity or of the temperature were available. At present, it is possible only to indicate the orders of magnitude of the several terms in the equations. It is probable that each of the three variables (current, salinity, temperature), is the sum of a number of periodic functions of the time, together with a part independent of t. The equations should therefore be applied to each of these component parts; but, in view of the scanty nature of the data, this is quite impracticable.

The equations have been applied numerically to the North Atlantic, taken as a whole, rather than to particular points in it, the aim, as stated above, being to obtain orders of magnitude only, and not definite numerical results. A very large number of observations of salinity and of temperature, spread over a period of two years, have been collected by H. N. Dickson*; and, from these, he has constructed charts showing the isohalines and the isotherms for each of the twenty-four months of the period covered. Their most striking feature is the difference between charts relating to the same

* 'Phil. Trans.,' A, vol. 196, p. 66 (1901).

month of two consecutive years. It is probable that this variation can be attributed to meteorological causes; for it is clear that such events as the breaking up of the polar ice will have a profound effect on the chemical and physical conditions in the Ocean.

Adopting the usual practice of measuring the salinity of the water by the number of parts by weight of salts dissolved in a thousand parts of water, we find as rough mean values* :—

$$s' \,(=)\, 0{\cdot}2 \text{ per 100 sea miles,}$$

$$\mathrm{T}' \,(=)\, 2^\circ \text{ C. per 100 sea miles,}$$

or using C.G.S. units,

$$s' \,(=)\, 3 \times 10^{-10}\, s, \qquad \mathrm{T}' \,(=)\, 2 \times 10^{-7}. \tag{13}$$

In order to obtain the time-gradients of the salinity from the charts, three representative points in latitude 50° N., with longitude 20°, 30°, 40° W., were taken, and for each the salinity-time curve was constructed by interpolation. The graphs naturally show some irregularities; but it is evident that the predominant parts have a period of six months, their amplitudes being about 0·1 for the first two stations, and 0·4 for the third. Taking 0·2 as the mean amplitude, we find for this term

$$\partial s/\partial t \,(=)\, 2 \times 10^{-9}\, s. \tag{14}$$

The effect of the very small daily variation in salinity is small compared with this, notwithstanding the more rapid rate of variation.

In a similar manner, the time-gradients of the temperature can be obtained. As could have been foreseen, the predominating term has a period of a year, its amplitude being about 3° C. in each case. This is approximately half the mean of the range of surface temperature of all the oceans. There is, in addition, a daily component, whose amplitude is about 0·3° C. This is, therefore, the more important part, and we may suppose

$$\partial \mathrm{T}/\partial t \,(=)\, 10^{-5}. \tag{15}$$

The vertical gradients of the salinity and the temperature are less easy to obtain. Thoulet† has calculated that the mean value of the salinity of the North Atlantic for depths less than 50 m. is 35·82, and for depths from 50 m. to 100 m. is 35·99. Using these figures, we find $\partial s/\partial z \,(=)\, 3 \times 10^{-5}$. Other observations confirm the accuracy of this estimate. Thus we may take

$$\partial s/\partial z \,(=)\, 10^{-6}\, s, \tag{16}$$

* The symbol (=) is used to denote equality of orders of magnitude.

† 'Résultats des Campagnes scientif. du Prince de Monaco,' vol. 29, p. 95.

while from a consideration of the vertical variation of temperature at many stations we infer

$$\partial T/\partial z\,(=)\,10^{-4}. \tag{17}$$

The most important part of the vertical velocity of the water particles at the surface of the ocean at points far removed from land may be supposed due to a semi-diurnal wave of amplitude of the order 30 cm. Hence

$$w\,(=)\,5 \times 10^{-3}. \tag{18}$$

From equations (13), (14), (16), (18) we see that the terms of (11) are of orders

$$2 \times 10^{-9}s, \qquad 3Q \times 10^{-10}s, \qquad 5 \times 10^{-9}s,$$

which are in the ratio

$$20, \qquad 3Q, \qquad 50.$$

Thus Q is of the order 20 cm. per sec. or, say, half a knot.

In a similar manner we see from (13), (15), (17), (18) that the terms of (12) are of orders

$$10^{-5}, \qquad 2Q \times 10^{-7}, \qquad 5 \times 10^{-7},$$

which are in the ratio

$$100, \qquad 2Q, \qquad 5.$$

Thus Q is of the order 50 cm. per sec. or one knot.

The mean current velocity of the surface of the Atlantic Ocean is rather less than a knot. The agreement of these two rough estimates with the observed result is satisfactory, and seems to confirm the mutual consistency of the data employed.